

High-Order Linear Block Algorithm for Numerical Integration of Second-Kind Volterra Integro-Differential Equations

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ABSTRACT

This study presents a new third-derivative Linear Block Algorithm (LBA) for the numerical integration of Volterra integro-differential equations of the second kind. Using the generalized linear multistep method and Computer Algebra System (CAS) support, a family of high-order block schemes was derived and expressed in continuous and discrete forms. The analytical properties of the new LBA scheme including order, error constant, consistency, zero-stability, convergence and region of absolute stability are rigorously examined. The new LBA scheme achieves a uniform order eight and demonstrates strong zero-stability, guaranteeing convergence according to Dahlquist's theorem. Furthermore, the region of absolute stability was sketched using the boundary locus method and it was found to be A-stable. Numerical experiments on both linear and nonlinear Volterra integro-differential equations reveal that the new LBA scheme produces highly accurate results, often matching exact solutions up to machine precision. When compared with established methods such as Adams-Bashforth-Moulton, Generalized Backward Differentiation Formulas, block multistep methods and classical quadrature rules, the new LBA scheme exhibits superior accuracy and enhanced stability across all numerical examples. These results confirm the efficiency, robustness and reliability of the newly developed third-derivative LBA scheme for solving Volterra integro-differential equations of second kind.

Keywords: Linear Block Algorithm (LBA); Third-Derivative Scheme; Volterra Integro-Differential Equations (VIDEs); Numerical Integration; High-Order Methods; Zero-Stability; Convergence Analysis; Boundary Locus Method; Error Constants; Multistep Block Methods; Stability Region; Computer Algebra System (CAS).

1. Introduction

This study considers the numerical integration of second-kind Volterra Integro-Differential Equations (VIDEs) of the form

$$\rho^\lambda(\xi) = \mathcal{G}(\xi) + \varphi \int_0^\xi K(\xi, \tau) \rho(\tau) d\tau \quad (1.1)$$

using a general linear multistep method of the form

$$\sum_{j=0}^1 \alpha_j \rho_{n+j} = h^\mu \sum_{j=0}^1 \beta_j \mathcal{G}_{n+j} \quad (1.2)$$

The study of second-kind Volterra Integro-Differential Equations (VIDEs) has attracted considerable attention due to their broad applications in science and engineering. Traditional solution techniques, including direct computation, Adomian decomposition, variational iteration, successive approximations and successive substitutions, often face challenges such as computational inefficiency for higher-order equations and difficulties in implementing unrealistic series (Wazwaz, 2015). To address these limitations, a numerical method based on the third- derivative Linear Block Algorithm (LBA) was developed, utilizing a Computer Algebra System (CAS).

The Adomian Decomposition Method (ADM), developed by George Adomian in the 1970s-1990s, is a semi-analytical technique for solving linear and nonlinear differential equations without linearization or perturbation. By decomposing complex problems into a convergent series of solvable components and using Adomian polynomials for nonlinear terms, ADM provides accurate, stable, and often closed-form approximate solutions across physics,

engineering and applied mathematics (Maturi & Malaikah, 2021; Mak et al., 2021; Chang, 2019). Recent advancements have focused on improving convergence and computational efficiency, including modified decomposition methods for Volterra-type of integral equations and applications in modeling real-world phenomena such as disease spread (Kareem et al., 2022; Agbata et al., 2022). Despite challenges such as sensitivity to initial conditions and computational complexity for higher-order polynomials, ADM remains a robust and versatile analytical tool (Islam & Srivastava, 2024a, 2024b).

Complementary numerical methods, including the Variational Iteration Method (VIM), Direct Computation Method (DCM) and Linear Block Approach (LBA), further expand the toolkit for solving differential and integral equations. VIM leverages Lagrange multipliers for rapidly converging iterative approximations without linearization, showing high efficiency in nonlinear problems, though it depends heavily on initial guesses and transformation of integral equations (Memon et al., 2023; Liu et al., 2025; Islam & Srivastava, 2024a). DCM offers a direct, non-iterative approach using numerical quadrature, particularly effective for linear or separable kernels, and has inspired hybrid and transform-based schemes to improve convergence and precision (Oyedepo et al., 2024; Adu et al., 2023; Aggarwal et al., 2023). The LBA, developed by Adeyeye & Omar (2016), provides efficient multi-point computation for initial value problems, handling stiff and higher-order systems with superior accuracy and stability, further enhanced by generalized multi-step algorithms (Sabo & Adeyeye, 2025; Raymond et al., 2023). Collectively, these methods illustrate the evolving landscape of numerical and semi-analytical techniques, balancing accuracy, computational efficiency and adaptability across scientific and engineering applications.

1.1. Objectives of the Study

The following are objectives of the study:

1. To derive some third-derivative numerical schemes using the LBA.
2. To obtain the continuous form of third derivative numerical schemes.
3. To obtain the higher order numerical schemes using the LBA.
4. To establish the basic properties of the new numerical schemes.
5. To apply the numerical schemes on some Volterra integral equations of second kind.
6. To compare the numerical results obtain with the existing methods.

2. Derivation of the New Linear Block Algorithm (LBA) Scheme

The new third derivative Linear Block Algorithm (LBA) is derived with the help of proposition 2.1 according to Sabo & Adeyeye (2025).

Proposition 2.1

The general linear multistep method (1.2) only one numerical scheme exists from every one-step block hybrid method. The linear block algorithm of the form

$$\rho_{n+\eta} = \sum_{j=0}^2 \frac{(\eta h)^j}{j!} \rho_n^{(j)} + \sum_{j=0}^1 (\Lambda_{j\eta} \mathcal{G}_{n+j}), \quad \eta = -\frac{1}{5}, -\frac{2}{5}, 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1 \quad (2.1)$$

and the higher derivatives of (2.1) of the form

$$\rho_{n+\eta}^{\sigma} = \sum_{j=0}^{2-\tau} \frac{(\eta h)^j}{j!} \rho_n^{(j+\sigma)} + \sum_{j=0}^7 (X_{j\sigma} \mathcal{G}_{n+j}) \quad \sigma=1 \left(-\frac{1}{5}, -\frac{2}{5}, 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1 \right), \quad \sigma=2 \left(-\frac{1}{5}, -\frac{2}{5}, 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1 \right) \quad (2.2)$$

is consider, where $\Lambda_{\eta j} = \Psi^{-1}Z$, $X_{\eta j\sigma} = \Psi^{-1}E$ and

$$\Psi = \begin{pmatrix} \frac{1}{(-\frac{1}{5}h)^1} & \frac{1}{(-\frac{2}{5}h)^1} & \frac{1}{(0)^1} & \frac{1}{(\frac{1}{5}h)^1} & \frac{1}{(\frac{2}{5}h)^1} & \frac{1}{(\frac{3}{5}h)^1} & \frac{1}{(\frac{4}{5}h)^1} & \frac{1}{(h)^1} \\ \frac{1!}{(-\frac{1}{5}h)^2} & \frac{1!}{(-\frac{2}{5}h)^2} & \frac{1!}{(0)^2} & \frac{1!}{(\frac{1}{5}h)^2} & \frac{1!}{(\frac{2}{5}h)^2} & \frac{1!}{(\frac{3}{5}h)^2} & \frac{1!}{(\frac{4}{5}h)^2} & \frac{1!}{(h)^2} \\ \frac{2!}{(-\frac{1}{5}h)^3} & \frac{2!}{(-\frac{2}{5}h)^3} & \frac{2!}{(0)^3} & \frac{2!}{(\frac{1}{5}h)^3} & \frac{2!}{(\frac{2}{5}h)^3} & \frac{2!}{(\frac{3}{5}h)^3} & \frac{2!}{(\frac{4}{5}h)^3} & \frac{2!}{(h)^3} \\ \frac{3!}{(-\frac{1}{5}h)^4} & \frac{3!}{(-\frac{2}{5}h)^4} & \frac{3!}{(0)^4} & \frac{3!}{(\frac{1}{5}h)^4} & \frac{3!}{(\frac{2}{5}h)^4} & \frac{3!}{(\frac{3}{5}h)^4} & \frac{3!}{(\frac{4}{5}h)^4} & \frac{3!}{(h)^4} \\ \frac{4!}{(-\frac{1}{5}h)^5} & \frac{4!}{(-\frac{2}{5}h)^5} & \frac{4!}{(0)^5} & \frac{4!}{(\frac{1}{5}h)^5} & \frac{4!}{(\frac{2}{5}h)^5} & \frac{4!}{(\frac{3}{5}h)^5} & \frac{4!}{(\frac{4}{5}h)^5} & \frac{4!}{(h)^5} \\ \frac{5!}{(-\frac{1}{5}h)^6} & \frac{5!}{(-\frac{2}{5}h)^6} & \frac{5!}{(0)^6} & \frac{5!}{(\frac{1}{5}h)^6} & \frac{5!}{(\frac{2}{5}h)^6} & \frac{5!}{(\frac{3}{5}h)^6} & \frac{5!}{(\frac{4}{5}h)^6} & \frac{5!}{(h)^6} \\ \frac{6!}{(-\frac{1}{5}h)^7} & \frac{6!}{(-\frac{2}{5}h)^7} & \frac{6!}{(0)^7} & \frac{6!}{(\frac{1}{5}h)^7} & \frac{6!}{(\frac{2}{5}h)^7} & \frac{6!}{(\frac{3}{5}h)^7} & \frac{6!}{(\frac{4}{5}h)^7} & \frac{6!}{(h)^7} \end{pmatrix}, \quad Z = \begin{pmatrix} \frac{(\xi h)^3}{3!} \\ \frac{(\xi h)^4}{4!} \\ \frac{(\xi h)^5}{5!} \\ \frac{(\xi h)^6}{6!} \\ \frac{(\xi h)^7}{7!} \\ \frac{(\xi h)^8}{8!} \\ \frac{(\xi h)^9}{9!} \\ \frac{(\xi h)^{10}}{10!} \end{pmatrix}, \quad E = \begin{pmatrix} \frac{(\xi h)^{3-\sigma}}{(3-\sigma)!} \\ \frac{(\xi h)^{4-\sigma}}{(4-\sigma)!} \\ \frac{(\xi h)^{5-\sigma}}{(5-\sigma)!} \\ \frac{(\xi h)^{6-\sigma}}{(6-\sigma)!} \\ \frac{(\xi h)^{7-\sigma}}{(7-\sigma)!} \\ \frac{(\xi h)^{8-\sigma}}{(8-\sigma)!} \\ \frac{(\xi h)^{9-\sigma}}{(9-\sigma)!} \\ \frac{(\xi h)^{10-\sigma}}{(10-\sigma)!} \end{pmatrix}$$

Proof

In order to obtain a new third-derivative numerical scheme with six grid points, equations (2.1) and (2.2) are solved sequentially to derive a polynomial of the form

$$\rho(\xi_n + \eta h) = \alpha_1 \rho_{\frac{n+1}{5}} + \alpha_2 \rho_{\frac{n+2}{5}} + \alpha_3 \rho_{\frac{n+3}{5}} + h^3 \left(\beta_{-\frac{1}{5}} \mathcal{G}_{n-\frac{1}{5}} + \beta_{-\frac{2}{5}} \mathcal{G}_{n-\frac{2}{5}} + \beta_0 \mathcal{G}_n + \beta_{\frac{1}{5}} \mathcal{G}_{n+\frac{1}{5}} + \beta_{\frac{2}{5}} \mathcal{G}_{n+\frac{2}{5}} + \beta_{\frac{3}{5}} \mathcal{G}_{n+\frac{3}{5}} + \beta_{\frac{4}{5}} \mathcal{G}_{n+\frac{4}{5}} + \beta_1 \mathcal{G}_{n+1} \right) \quad (2.3)$$

Where $\eta = \xi_n + \xi h$ and

$$\left. \begin{aligned} \alpha_{\frac{1}{5}} &= 3 - \frac{25}{2}\eta + \frac{25}{2}\eta^2, \quad \alpha_{\frac{2}{5}} = -3 + 20\eta - 25\eta^2, \quad \alpha_{\frac{3}{5}} = 1 - \frac{15}{2}\eta + \frac{25}{2}\eta^2, \\ \beta_{-\frac{1}{5}} &= -\frac{281}{7560000} - \frac{1441}{45360000}\eta + \frac{211}{90720}\eta^2 - \frac{5}{72}\eta^4 + \frac{107}{432}\eta^5 - \frac{55}{216}\eta^6 - \frac{1375}{6048}\eta^7 + \frac{34375}{48384}\eta^8 - \frac{40625}{72576}\eta^9 + \frac{3125}{20736}\eta^{10} \\ \beta_{-\frac{2}{5}} &= \frac{31}{7560000} - \frac{133}{22680000}\eta - \frac{645}{3628800}\eta^2 + \frac{5}{1008}\eta^4 - \frac{11}{864}\eta^5 - \frac{35}{3456}\eta^6 + \frac{125}{1512}\eta^7 - \frac{3125}{24192}\eta^8 + \frac{3125}{36288}\eta^9 - \frac{3125}{145152}\eta^{10} \\ \beta_0 &= \frac{73}{840000} + \frac{617}{210000}\eta - \frac{2017}{48384}\eta^2 + \frac{1}{6}\eta^4 - \frac{47}{288}\eta^5 - \frac{7}{16}\eta^6 + \frac{385}{384}\eta^7 - \frac{625}{384}\eta^8 + \frac{3125}{2016}\eta^9 - \frac{3125}{6912}\eta^{10} \\ \beta_{\frac{1}{5}} &= -\frac{30647}{7560000} + \frac{371773}{9072000}\eta - \frac{52907}{453600}\eta^2 + \frac{25}{72}\eta^4 + \frac{65}{432}\eta^5 - \frac{625}{432}\eta^6 + \frac{3875}{6048}\eta^7 + \frac{96875}{48384}\eta^8 - \frac{171875}{72576}\eta^9 + \frac{15625}{20736}\eta^{10} \\ \beta_{\frac{2}{5}} &= -\frac{30647}{7560000} + \frac{19121}{648000}\eta - \frac{149993}{3628800}\eta^2 - \frac{25}{144}\eta^4 + \frac{85}{864}\eta^5 + \frac{3575}{3456}\eta^6 - \frac{1375}{1512}\eta^7 - \frac{34375}{24192}\eta^8 + \frac{78125}{36288}\eta^9 - \frac{15625}{20736}\eta^{10} \\ \beta_{\frac{3}{5}} &= \frac{73}{840000} - \frac{1031}{5040000}\eta - \frac{2}{675}\eta^2 + \frac{5}{72}\eta^4 - \frac{1}{16}\eta^5 - \frac{5}{12}\eta^6 + \frac{125}{224}\eta^7 + \frac{3125}{5376}\eta^8 - \frac{3125}{2688}\eta^9 + \frac{3125}{6912}\eta^{10} \\ \beta_{\frac{4}{5}} &= -\frac{281}{7560000} + \frac{1889}{11340000}\eta + \frac{1943}{3628800}\eta^2 - \frac{5}{288}\eta^4 + \frac{1}{54}\eta^5 + \frac{355}{3456}\eta^6 - \frac{125}{756}\eta^7 - \frac{3125}{24192}\eta^8 + \frac{3125}{9072}\eta^9 - \frac{3125}{20736}\eta^{10} \\ \beta_1 &= \frac{31}{7560000} - \frac{817}{45360000}\eta - \frac{1}{16200}\eta^2 + \frac{1}{504}\eta^4 - \frac{1}{432}\eta^5 - \frac{5}{432}\eta^6 + \frac{125}{6048}\eta^7 + \frac{625}{48384}\eta^8 - \frac{3125}{72576}\eta^9 + \frac{3125}{145152}\eta^{10} \end{aligned} \right\} \quad (2.4)$$

The generalized algorithm (2.1) is expanded to give a new numerical scheme as

$$\left. \begin{aligned} \rho_{n-\frac{1}{5}} &= \rho_n - \frac{1}{5}h\rho'_n + \frac{\left(-\frac{1}{5}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{10}\vartheta_{n-\frac{1}{5}} + \Lambda_{11}\vartheta_{n-\frac{2}{5}} + \Lambda_{12}\vartheta_n + \Lambda_{13}\vartheta_{n+\frac{1}{5}} + \Lambda_{14}\vartheta_{n+\frac{2}{5}} + \Lambda_{15}\vartheta_{n+\frac{3}{5}} + \Lambda_{16}\vartheta_{n+\frac{4}{5}} + \Lambda_{17}\vartheta_{n+1}\right) \\ \rho_{n-\frac{2}{5}} &= \rho_n - \frac{2}{5}h\rho'_n + \frac{\left(-\frac{2}{5}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{20}\vartheta_{n-\frac{1}{5}} + \Lambda_{21}\vartheta_{n-\frac{2}{5}} + \Lambda_{22}\vartheta_n + \Lambda_{23}\vartheta_{n+\frac{1}{5}} + \Lambda_{24}\vartheta_{n+\frac{2}{5}} + \Lambda_{25}\vartheta_{n+\frac{3}{5}} + \Lambda_{26}\vartheta_{n+\frac{4}{5}} + \Lambda_{27}\vartheta_{n+1}\right) \\ \rho_{n+\frac{1}{5}} &= \rho_n + \frac{1}{5}h\rho'_n + \frac{\left(\frac{1}{5}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{30}\vartheta_{n-\frac{1}{5}} + \Lambda_{31}\vartheta_{n-\frac{2}{5}} + \Lambda_{32}\vartheta_n + \Lambda_{33}\vartheta_{n+\frac{1}{5}} + \Lambda_{34}\vartheta_{n+\frac{2}{5}} + \Lambda_{35}\vartheta_{n+\frac{3}{5}} + \Lambda_{36}\vartheta_{n+\frac{4}{5}} + \Lambda_{37}\vartheta_{n+1}\right) \\ \rho_{n+\frac{2}{5}} &= \rho_n + \frac{2}{5}h\rho'_n + \frac{\left(\frac{2}{5}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{40}\vartheta_{n-\frac{1}{5}} + \Lambda_{41}\vartheta_{n-\frac{2}{5}} + \Lambda_{42}\vartheta_n + \Lambda_{43}\vartheta_{n+\frac{1}{5}} + \Lambda_{44}\vartheta_{n+\frac{2}{5}} + \Lambda_{45}\vartheta_{n+\frac{3}{5}} + \Lambda_{46}\vartheta_{n+\frac{4}{5}} + \Lambda_{47}\vartheta_{n+1}\right) \\ \rho_{n+\frac{3}{5}} &= \rho_n + \frac{3}{5}h\rho'_n + \frac{\left(\frac{3}{5}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{50}\vartheta_{n-\frac{1}{5}} + \Lambda_{51}\vartheta_{n-\frac{2}{5}} + \Lambda_{52}\vartheta_n + \Lambda_{53}\vartheta_{n+\frac{1}{5}} + \Lambda_{54}\vartheta_{n+\frac{2}{5}} + \Lambda_{55}\vartheta_{n+\frac{3}{5}} + \Lambda_{56}\vartheta_{n+\frac{4}{5}} + \Lambda_{57}\vartheta_{n+1}\right) \\ \rho_{n+\frac{4}{5}} &= \rho_n + \frac{4}{5}h\rho'_n + \frac{\left(\frac{4}{5}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{60}\vartheta_{n-\frac{1}{5}} + \Lambda_{61}\vartheta_{n-\frac{2}{5}} + \Lambda_{62}\vartheta_n + \Lambda_{63}\vartheta_{n+\frac{1}{5}} + \Lambda_{64}\vartheta_{n+\frac{2}{5}} + \Lambda_{65}\vartheta_{n+\frac{3}{5}} + \Lambda_{66}\vartheta_{n+\frac{4}{5}} + \Lambda_{67}\vartheta_{n+1}\right) \\ \rho_{n+1} &= \rho_n + h\rho'_n + \frac{(h)^2}{2!}\rho''_n + h^3\left(\Lambda_{70}\vartheta_{n-\frac{1}{5}} + \Lambda_{71}\vartheta_{n-\frac{2}{5}} + \Lambda_{72}\vartheta_n + \Lambda_{73}\vartheta_{n+\frac{1}{5}} + \Lambda_{74}\vartheta_{n+\frac{2}{5}} + \Lambda_{75}\vartheta_{n+\frac{3}{5}} + \Lambda_{76}\vartheta_{n+\frac{4}{5}} + \Lambda_{77}\vartheta_{n+1}\right) \end{aligned} \right\} \quad (2.5)$$

The higher derivatives of the new numerical scheme are

$$\left. \begin{aligned} \rho'_{n-\frac{1}{5}} &= \rho'_n - \frac{1}{5}h\rho''_n + h^2\left(X_{110}\vartheta_{n-\frac{1}{5}} + X_{111}\vartheta_{n-\frac{2}{5}} + X_{112}\vartheta_n + X_{113}\vartheta_{n+\frac{1}{5}} + X_{114}\vartheta_{n+\frac{2}{5}} + X_{115}\vartheta_{n+\frac{3}{5}} + X_{116}\vartheta_{n+\frac{4}{5}} + X_{117}\vartheta_{n+1}\right) \\ \rho'_{n-\frac{2}{5}} &= \rho'_n - \frac{2}{5}h\rho''_n + h^2\left(X_{120}\vartheta_{n-\frac{1}{5}} + X_{121}\vartheta_{n-\frac{2}{5}} + X_{122}\vartheta_n + X_{123}\vartheta_{n+\frac{1}{5}} + X_{124}\vartheta_{n+\frac{2}{5}} + X_{125}\vartheta_{n+\frac{3}{5}} + X_{126}\vartheta_{n+\frac{4}{5}} + X_{127}\vartheta_{n+1}\right) \\ \rho'_{n+\frac{1}{5}} &= \rho'_n + \frac{1}{5}h\rho''_n + h^2\left(X_{130}\vartheta_{n-\frac{1}{5}} + X_{131}\vartheta_{n-\frac{2}{5}} + X_{132}\vartheta_n + X_{133}\vartheta_{n+\frac{1}{5}} + X_{134}\vartheta_{n+\frac{2}{5}} + X_{135}\vartheta_{n+\frac{3}{5}} + X_{136}\vartheta_{n+\frac{4}{5}} + X_{137}\vartheta_{n+1}\right) \\ \rho'_{n+\frac{2}{5}} &= \rho'_n + \frac{2}{5}h\rho''_n + h^2\left(X_{140}\vartheta_{n-\frac{1}{5}} + X_{141}\vartheta_{n-\frac{2}{5}} + X_{142}\vartheta_n + X_{143}\vartheta_{n+\frac{1}{5}} + X_{144}\vartheta_{n+\frac{2}{5}} + X_{145}\vartheta_{n+\frac{3}{5}} + X_{146}\vartheta_{n+\frac{4}{5}} + X_{147}\vartheta_{n+1}\right) \\ \rho'_{n+\frac{3}{5}} &= \rho'_n + \frac{3}{5}h\rho''_n + h^2\left(X_{150}\vartheta_{n-\frac{1}{5}} + X_{151}\vartheta_{n-\frac{2}{5}} + X_{152}\vartheta_n + X_{153}\vartheta_{n+\frac{1}{5}} + X_{154}\vartheta_{n+\frac{2}{5}} + X_{155}\vartheta_{n+\frac{3}{5}} + X_{156}\vartheta_{n+\frac{4}{5}} + X_{157}\vartheta_{n+1}\right) \\ \rho'_{n+\frac{4}{5}} &= \rho'_n + \frac{4}{5}h\rho''_n + h^2\left(X_{160}\vartheta_{n-\frac{1}{5}} + X_{161}\vartheta_{n-\frac{2}{5}} + X_{162}\vartheta_n + X_{163}\vartheta_{n+\frac{1}{5}} + X_{164}\vartheta_{n+\frac{2}{5}} + X_{165}\vartheta_{n+\frac{3}{5}} + X_{166}\vartheta_{n+\frac{4}{5}} + X_{167}\vartheta_{n+1}\right) \\ \rho'_{n+1} &= \rho'_n + h\rho''_n + h^2\left(X_{170}\vartheta_{n-\frac{1}{5}} + X_{171}\vartheta_{n-\frac{2}{5}} + X_{172}\vartheta_n + X_{173}\vartheta_{n+\frac{1}{5}} + X_{174}\vartheta_{n+\frac{2}{5}} + X_{175}\vartheta_{n+\frac{3}{5}} + X_{176}\vartheta_{n+\frac{4}{5}} + X_{177}\vartheta_{n+1}\right) \end{aligned} \right\} \quad (2.6)$$

$$\left. \begin{aligned} \rho''_{n-\frac{1}{5}} &= \rho''_n + h\left(X_{210}\vartheta_{n-\frac{1}{5}} + X_{211}\vartheta_{n-\frac{2}{5}} + X_{212}\vartheta_n + X_{213}\vartheta_{n+\frac{1}{5}} + X_{214}\vartheta_{n+\frac{2}{5}} + X_{215}\vartheta_{n+\frac{3}{5}} + X_{216}\vartheta_{n+\frac{4}{5}} + X_{217}\vartheta_{n+1}\right) \\ \rho''_{n-\frac{2}{5}} &= \rho''_n + h\left(X_{220}\vartheta_{n-\frac{1}{5}} + X_{221}\vartheta_{n-\frac{2}{5}} + X_{222}\vartheta_n + X_{223}\vartheta_{n+\frac{1}{5}} + X_{224}\vartheta_{n+\frac{2}{5}} + X_{225}\vartheta_{n+\frac{3}{5}} + X_{226}\vartheta_{n+\frac{4}{5}} + X_{227}\vartheta_{n+1}\right) \\ \rho''_{n+\frac{1}{5}} &= \rho''_n + h\left(X_{230}\vartheta_{n-\frac{1}{5}} + X_{231}\vartheta_{n-\frac{2}{5}} + X_{232}\vartheta_n + X_{233}\vartheta_{n+\frac{1}{5}} + X_{234}\vartheta_{n+\frac{2}{5}} + X_{235}\vartheta_{n+\frac{3}{5}} + X_{236}\vartheta_{n+\frac{4}{5}} + X_{237}\vartheta_{n+1}\right) \\ \rho''_{n+\frac{2}{5}} &= \rho''_n + h\left(X_{240}\vartheta_{n-\frac{1}{5}} + X_{241}\vartheta_{n-\frac{2}{5}} + X_{242}\vartheta_n + X_{243}\vartheta_{n+\frac{1}{5}} + X_{244}\vartheta_{n+\frac{2}{5}} + X_{245}\vartheta_{n+\frac{3}{5}} + X_{246}\vartheta_{n+\frac{4}{5}} + X_{247}\vartheta_{n+1}\right) \\ \rho''_{n+\frac{3}{5}} &= \rho''_n + h\left(X_{250}\vartheta_{n-\frac{1}{5}} + X_{251}\vartheta_{n-\frac{2}{5}} + X_{252}\vartheta_n + X_{253}\vartheta_{n+\frac{1}{5}} + X_{254}\vartheta_{n+\frac{2}{5}} + X_{255}\vartheta_{n+\frac{3}{5}} + X_{256}\vartheta_{n+\frac{4}{5}} + X_{257}\vartheta_{n+1}\right) \\ \rho''_{n+\frac{4}{5}} &= \rho''_n + h\left(X_{260}\vartheta_{n-\frac{1}{5}} + X_{261}\vartheta_{n-\frac{2}{5}} + X_{262}\vartheta_n + X_{263}\vartheta_{n+\frac{1}{5}} + X_{264}\vartheta_{n+\frac{2}{5}} + X_{265}\vartheta_{n+\frac{3}{5}} + X_{266}\vartheta_{n+\frac{4}{5}} + X_{267}\vartheta_{n+1}\right) \\ \rho''_{n+1} &= \rho''_n + h\left(X_{270}\vartheta_{n-\frac{1}{5}} + X_{271}\vartheta_{n-\frac{2}{5}} + X_{272}\vartheta_n + X_{273}\vartheta_{n+\frac{1}{5}} + X_{274}\vartheta_{n+\frac{2}{5}} + X_{275}\vartheta_{n+\frac{3}{5}} + X_{276}\vartheta_{n+\frac{4}{5}} + X_{277}\vartheta_{n+1}\right) \end{aligned} \right\} \quad (2.7)$$

By simplifying $\Lambda_{\eta,j} = \Psi^{-1}Z$, the unknown coefficients of Λ in equation (2.5) are obtained as

$$\begin{pmatrix} \Lambda_{10} \\ \Lambda_{11} \\ \Lambda_{12} \\ \Lambda_{13} \\ \Lambda_{14} \\ \Lambda_{15} \\ \Lambda_{16} \\ \Lambda_{17} \end{pmatrix} = \begin{pmatrix} 6533 \\ 3240000 \\ 1501 \\ 15120000 \\ 210967 \\ 15120000 \\ 1249 \\ 3024000 \\ 21431 \\ 9072000 \\ 7517 \\ 75600000 \\ 143 \\ 5600000 \\ 673 \\ 226800000 \end{pmatrix}, \begin{pmatrix} \Lambda_{20} \\ \Lambda_{21} \\ \Lambda_{22} \\ \Lambda_{23} \\ \Lambda_{24} \\ \Lambda_{25} \\ \Lambda_{26} \\ \Lambda_{27} \end{pmatrix} = \begin{pmatrix} 2573 \\ 590625 \\ 103 \\ 3543750 \\ 4594 \\ 590625 \\ 169 \\ 70875 \\ 307 \\ 236250 \\ 313 \\ 590625 \\ 236 \\ 1771875 \\ 1 \\ 65625 \end{pmatrix}, \begin{pmatrix} \Lambda_{30} \\ \Lambda_{31} \\ \Lambda_{32} \\ \Lambda_{33} \\ \Lambda_{34} \\ \Lambda_{35} \\ \Lambda_{36} \\ \Lambda_{37} \end{pmatrix} = \begin{pmatrix} 3743 \\ 75600000 \\ 1807 \\ 453600000 \\ 150137 \\ 151200000 \\ 3391 \\ 6480000 \\ 635 \\ 3360000 \\ 5483 \\ 75600000 \\ 7967 \\ 453600000 \\ 149 \\ 75600000 \end{pmatrix}, \begin{pmatrix} \Lambda_{40} \\ \Lambda_{41} \\ \Lambda_{42} \\ \Lambda_{43} \\ \Lambda_{44} \\ \Lambda_{45} \\ \Lambda_{46} \\ \Lambda_{47} \end{pmatrix} = \begin{pmatrix} 571 \\ 1771875 \\ 31 \\ 1181250 \\ 3194 \\ 590625 \\ 83 \\ 13125 \\ 23 \\ 20250 \\ 277 \\ 590625 \\ 68 \\ 590625 \\ 23 \\ 1771875 \end{pmatrix}, \begin{pmatrix} \Lambda_{50} \\ \Lambda_{51} \\ \Lambda_{52} \\ \Lambda_{53} \\ \Lambda_{54} \\ \Lambda_{55} \\ \Lambda_{56} \\ \Lambda_{57} \end{pmatrix} = \begin{pmatrix} 2187 \\ 2800000 \\ 351 \\ 5600000 \\ 73683 \\ 5600000 \\ 2403 \\ 112000 \\ 1377 \\ 1120000 \\ 441 \\ 400000 \\ 1431 \\ 5600000 \\ 81 \\ 2800000 \end{pmatrix}, \begin{pmatrix} \Lambda_{60} \\ \Lambda_{61} \\ \Lambda_{62} \\ \Lambda_{63} \\ \Lambda_{64} \\ \Lambda_{65} \\ \Lambda_{66} \\ \Lambda_{67} \end{pmatrix} = \begin{pmatrix} 23 \\ 21875 \\ 208 \\ 1771875 \\ 14408 \\ 590625 \\ 3232 \\ 70875 \\ 1328 \\ 118125 \\ 3424 \\ 590625 \\ 104 \\ 253125 \\ 32 \\ 590625 \end{pmatrix}, \begin{pmatrix} \Lambda_{70} \\ \Lambda_{71} \\ \Lambda_{72} \\ \Lambda_{73} \\ \Lambda_{74} \\ \Lambda_{75} \\ \Lambda_{76} \\ \Lambda_{77} \end{pmatrix} = \begin{pmatrix} 169 \\ 72576 \\ 1 \\ 5376 \\ 1885 \\ 48384 \\ 1915 \\ 24192 \\ 4105 \\ 145152 \\ 463 \\ 24192 \\ 151 \\ 48384 \\ 11 \\ 72576 \end{pmatrix}$$

Similarly, the expressions $X_{\eta,j\sigma} = \Psi^{-1}E$ are simplified to determine the unknown coefficients of the higher derivative X in equations (2.6) and (2.7) as

$$\begin{pmatrix} X_{110} \\ X_{111} \\ X_{112} \\ X_{113} \\ X_{114} \\ X_{115} \\ X_{116} \\ X_{117} \end{pmatrix} = \begin{pmatrix} 7297 \\ 1620000 \\ 8563 \\ 45360000 \\ 302429 \\ 15120000 \\ 32233 \\ 4536000 \\ 37631 \\ 9072000 \\ 208 \\ 118125 \\ 20609 \\ 45360000 \\ 1201 \\ 22680000 \end{pmatrix}, \begin{pmatrix} X_{120} \\ X_{121} \\ X_{122} \\ X_{123} \\ X_{124} \\ X_{125} \\ X_{126} \\ X_{127} \end{pmatrix} = \begin{pmatrix} 16159 \\ 354375 \\ 119 \\ 50625 \\ 8467 \\ 236250 \\ 323 \\ 70875 \\ 2 \\ 2835 \\ 19 \\ 118125 \\ 83 \\ 708750 \\ 1 \\ 50625 \end{pmatrix}, \begin{pmatrix} X_{130} \\ X_{131} \\ X_{132} \\ X_{133} \\ X_{134} \\ X_{135} \\ X_{136} \\ X_{137} \end{pmatrix} = \begin{pmatrix} 17483 \\ 22680000 \\ 409 \\ 6480000 \\ 197611 \\ 15120000 \\ 3233 \\ 324000 \\ 29843 \\ 9072000 \\ 9127 \\ 7560000 \\ 13169 \\ 45360000 \\ 23 \\ 708750 \end{pmatrix}, \begin{pmatrix} X_{140} \\ X_{141} \\ X_{142} \\ X_{143} \\ X_{144} \\ X_{145} \\ X_{146} \\ X_{147} \end{pmatrix} = \begin{pmatrix} 649 \\ 354375 \\ 52 \\ 354375 \\ 7193 \\ 236250 \\ 3677 \\ 70875 \\ 23 \\ 10125 \\ 251 \\ 118125 \\ 397 \\ 708750 \\ 23 \\ 354375 \end{pmatrix}, \begin{pmatrix} X_{150} \\ X_{151} \\ X_{152} \\ X_{153} \\ X_{154} \\ X_{155} \\ X_{156} \\ X_{157} \end{pmatrix} = \begin{pmatrix} 201 \\ 70000 \\ 129 \\ 560000 \\ 26619 \\ 560000 \\ 5493 \\ 56000 \\ 168 \\ 22400 \\ 147 \\ 20000 \\ 627 \\ 560000 \\ 33 \\ 280000 \end{pmatrix}, \begin{pmatrix} X_{160} \\ X_{161} \\ X_{162} \\ X_{163} \\ X_{164} \\ X_{165} \\ X_{166} \\ X_{167} \end{pmatrix} = \begin{pmatrix} 1328 \\ 354375 \\ 104 \\ 354375 \\ 7568 \\ 118125 \\ 10288 \\ 70875 \\ 4712 \\ 70875 \\ 5392 \\ 118125 \\ 104 \\ 50625 \\ 16 \\ 354375 \end{pmatrix}, \begin{pmatrix} X_{170} \\ X_{171} \\ X_{172} \\ X_{173} \\ X_{174} \\ X_{175} \\ X_{176} \\ X_{177} \end{pmatrix} = \begin{pmatrix} 205 \\ 36288 \\ 5 \\ 10368 \\ 2039 \\ 24192 \\ 1675 \\ 9072 \\ 8125 \\ 72576 \\ 965 \\ 12096 \\ 3035 \\ 72576 \\ 7 \\ 2592 \end{pmatrix}$$

$$\begin{pmatrix} X_{210} \\ X_{211} \\ X_{212} \\ X_{213} \\ X_{214} \\ X_{215} \\ X_{216} \\ X_{217} \end{pmatrix} = \begin{pmatrix} 5311 \\ 67200 \\ 55 \\ 24192 \\ 11261 \\ 67200 \\ 44797 \\ 604800 \\ 2987 \\ 67200 \\ 1283 \\ 67200 \\ 2999 \\ 604800 \\ 13 \\ 22400 \end{pmatrix}, \begin{pmatrix} X_{220} \\ X_{221} \\ X_{222} \\ X_{223} \\ X_{224} \\ X_{225} \\ X_{226} \\ X_{227} \end{pmatrix} = \begin{pmatrix} 1466 \\ 4725 \\ 41 \\ 700 \\ 71 \\ 2100 \\ 68 \\ 525 \\ 1927 \\ 18900 \\ 56 \\ 525 \\ 29 \\ 2100 \\ 8 \\ 4725 \end{pmatrix}, \begin{pmatrix} X_{230} \\ X_{231} \\ X_{232} \\ X_{233} \\ X_{234} \\ X_{235} \\ X_{236} \\ X_{237} \end{pmatrix} = \begin{pmatrix} 4183 \\ 604800 \\ 13 \\ 22400 \\ 6403 \\ 67200 \\ 9077 \\ 67200 \\ 20227 \\ 604800 \\ 803 \\ 67200 \\ 191 \\ 67200 \\ 191 \\ 604800 \end{pmatrix}, \begin{pmatrix} X_{240} \\ X_{241} \\ X_{242} \\ X_{243} \\ X_{244} \\ X_{245} \\ X_{246} \\ X_{247} \end{pmatrix} = \begin{pmatrix} 2 \\ 525 \\ 1 \\ 3780 \\ 167 \\ 2100 \\ 1172 \\ 4725 \\ 167 \\ 2100 \\ 2 \\ 525 \\ 1 \\ 3780 \\ 0 \end{pmatrix}, \begin{pmatrix} X_{250} \\ X_{251} \\ X_{252} \\ X_{253} \\ X_{254} \\ X_{255} \\ X_{256} \\ X_{257} \end{pmatrix} = \begin{pmatrix} 149 \\ 22400 \\ 13 \\ 22400 \\ 2049 \\ 22400 \\ 4807 \\ 22400 \\ 4807 \\ 22400 \\ 2049 \\ 22400 \\ 149 \\ 22400 \\ 13 \\ 22400 \end{pmatrix}, \begin{pmatrix} X_{260} \\ X_{261} \\ X_{262} \\ X_{263} \\ X_{264} \\ X_{265} \\ X_{266} \\ X_{267} \end{pmatrix} = \begin{pmatrix} 8 \\ 4725 \\ 0 \\ 525 \\ 138 \\ 525 \\ 664 \\ 4725 \\ 136 \\ 525 \\ 38 \\ 525 \\ 8 \\ 4725 \end{pmatrix}, \begin{pmatrix} X_{270} \\ X_{271} \\ X_{272} \\ X_{273} \\ X_{274} \\ X_{275} \\ X_{276} \\ X_{277} \end{pmatrix} = \begin{pmatrix} 55 \\ 2688 \\ 55 \\ 24192 \\ 379 \\ 2688 \\ 2725 \\ 24192 \\ 925 \\ 2688 \\ 155 \\ 2688 \\ 7345 \\ 24192 \\ 53 \\ 896 \end{pmatrix}$$

3. Analysis of new LBA Scheme

The new third-derivative LBA were numerically analyzed. These analyses are order and error constants, consistency, Zero-stability, Convergent and region of absolute stability.

3.1. Order and Error Constant of LBA Scheme

Using the corollary 3.1 and corollary 3.2 to obtained the order and error constant of new LBA scheme.

Corollary 3.1

The linear operator $L[\rho(\xi_n); h]$ associate with the local truncation error of the LBA defined in (2.5) and its higher derivatives (2.6) and (2.7) is given as

$$C_{08} h^{08} \rho^{08}(\xi_n) + O(h^{11}), C_{08} h^{08} \rho^{08}(\xi_n) + O(h^{10}), C_{08} h^{08} \rho^{08}(\xi_n) + O(h^{09}).$$

Corollary 3.2

The local truncation error of (2.5) to (2.7) assume $\rho(\xi)$ to be sufficiently differentiable and expanding equation (3.1) to (3.3) about ξ_n using a Taylor series to obtain

$$\begin{aligned} L_{-\frac{1}{5}}[\rho(\xi_n); h] &= (-5.0968 \times 10^{-12}), L_{-\frac{2}{5}}[\rho(\xi_n); h] = (-2.3806 \times 10^{-11}), L_{\frac{1}{5}}[\rho(\xi_n); h] = (-2.9455 \times 10^{-12}) \\ L_{\frac{2}{5}}[\rho(\xi_n); h] &= (-1.9538 \times 10^{-11}), L_{\frac{3}{5}}[\rho(\xi_n); h] = (-4.4883 \times 10^{-11}), L_{\frac{4}{5}}[\rho(\xi_n); h] = (-8.4586 \times 10^{-11}) \\ L_1[\rho(\xi_n); h] &= (-1.3308 \times 10^{-10}), \\ L_{-\frac{1}{5}}[\rho'(\xi_n); h] &= (9.2176 \times 10^{-11}), L_{-\frac{2}{5}}[\rho'(\xi_n); h] = (-9.3009 \times 10^{-11}), L_{\frac{1}{5}}[\rho'(\xi_n); h] = (-4.7845 \times 10^{-11}) \\ L_{\frac{2}{5}}[\rho'(\xi_n); h] &= (-1.0384 \times 10^{-10}), L_{\frac{3}{5}}[\rho'(\xi_n); h] = (-1.6800 \times 10^{-10}), L_{\frac{4}{5}}[\rho'(\xi_n); h] = (-1.8782 \times 10^{-11}) \\ L_1[\rho'(\xi_n); h] &= (-4.8501 \times 10^{-10}), \\ L_{-\frac{1}{5}}[\rho''(\xi_n); h] &= (-1.0296 \times 10^{-09}), L_{-\frac{2}{5}}[\rho''(\xi_n); h] = (3.7610 \times 10^{-09}), L_{\frac{1}{5}}[\rho''(\xi_n); h] = (-4.5616 \times 10^{-10}) \\ L_{\frac{2}{5}}[\rho''(\xi_n); h] &= (-1.0384 \times 10^{-10}), L_{\frac{3}{5}}[\rho''(\xi_n); h] = (-5.6000 \times 10^{-10}), L_{\frac{4}{5}}[\rho''(\xi_n); h] = (-4.6956 \times 10^{-10}) \\ L_1[\rho''(\xi_n); h] &= (-4.3210 \times 10^{-09}), \end{aligned}$$

Proof

Expanding equations (2.5) to (2.7) using Corollary 3.2 and collect the like terms to get

$$\begin{aligned} L_{-\frac{1}{5}}[\rho(\xi_n); h] &= (-5.0968 \times 10^{-12}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{11}), L_{-\frac{2}{5}}[\rho(\xi_n); h] = (-2.3806 \times 10^{-11}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{11}), \\ L_{\frac{1}{5}}[\rho(\xi_n); h] &= (-2.9455 \times 10^{-12}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{11}), L_{\frac{2}{5}}[\rho(\xi_n); h] = (-1.9538 \times 10^{-11}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{11}), \\ L_{\frac{3}{5}}[\rho(\xi_n); h] &= (-4.4883 \times 10^{-11}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{11}), L_{\frac{4}{5}}[\rho(\xi_n); h] = (-8.4586 \times 10^{-11}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{11}), \\ L_1[\rho(\xi_n); h] &= (-1.3308 \times 10^{-10}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{11}), \\ L_{-\frac{1}{5}}[\rho'(\xi_n); h] &= (9.2176 \times 10^{-11}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{10}), L_{-\frac{2}{5}}[\rho'(\xi_n); h] = (-9.3009 \times 10^{-11}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{10}), \\ L_{\frac{1}{5}}[\rho'(\xi_n); h] &= (-4.7845 \times 10^{-11}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{10}), L_{\frac{2}{5}}[\rho'(\xi_n); h] = (-1.0384 \times 10^{-10}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{10}), \\ L_{\frac{3}{5}}[\rho'(\xi_n); h] &= (-1.6800 \times 10^{-10}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{10}), L_{\frac{4}{5}}[\rho'(\xi_n); h] = (-1.8782 \times 10^{-11}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{10}), \\ L_1[\rho'(\xi_n); h] &= (-4.8501 \times 10^{-10}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{10}), \\ L_{-\frac{1}{5}}[\rho''(\xi_n); h] &= (-1.0296 \times 10^{-09}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{09}), L_{-\frac{2}{5}}[\rho''(\xi_n); h] = (3.7610 \times 10^{-09}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{09}), \\ L_{\frac{1}{5}}[\rho''(\xi_n); h] &= (-4.5616 \times 10^{-10}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{09}), L_{\frac{2}{5}}[\rho''(\xi_n); h] = (-1.0384 \times 10^{-10}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{09}), \\ L_{\frac{3}{5}}[\rho''(\xi_n); h] &= (-5.6000 \times 10^{-10}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{09}), L_{\frac{4}{5}}[\rho''(\xi_n); h] = (-4.6956 \times 10^{-10}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{09}), \\ L_1[\rho''(\xi_n); h] &= (-4.3210 \times 10^{-09}) C_{08} h^{08} \rho^{08}(\xi_n) + 0(h^{09}), \end{aligned}$$

3.2. Consistency of LBA Scheme

Definition 3.1: Given the new numerical scheme, the first and second characteristic polynomials are defined as,

$$\rho(z) = \sum_{j=0}^k \alpha_j z^j \quad (3.4)$$

$$\sigma(z) = \sum_{j=0}^k \beta_j z^j \quad (3.5)$$

where z is the principal root, $\alpha_k \neq 0$ and $\alpha_0^2 + \beta_0^2 \neq 0$. The new numerical scheme is said to be consistent if it satisfies the following conditions;

1) the order $p \geq 1$,

2) $\sum_{j=0}^k \alpha_j = 0$, and

3) $\rho'(1) = \sigma(1)$

According to definition 3.1, a LBA is consistent since it is of uniform order eight. Therefore, the LBA satisfies this condition and is deemed consistent.

3.3. Zero Stability of LBA Scheme

A LBA Scheme is said to be Zero-stable for any well behaved problems provided if

1) all roots of $\rho(r)$ lies in the unit disk, $|r| \leq 1$

2) any roots on the unit circle ($|r| = 1$) are simple.

Hence

$$\rho(u) = \frac{91674240 u - 1068480 u^2 + 4689496 u^3 + 46746 u^4 + 26397 u^5 + 7687598 u^6}{58060800 - 14515200 u + 604800 u^2 + 151200 u^3 - 15120 u^4 - 630 u^5 + 135 u^6 + 21457 u^7} \quad (3.6)$$

Therefore, equating (3.6) equal to zero and solving for u gives $u = 1$, hence the LBA scheme is zero-stable.

3.4. Convergence of LBA Scheme

By Dahlquist theorem, the necessary and sufficient conditions for LBA scheme to be convergent is that it must be consistent and zero-stable (Sabo & Adeyeye, 2025). Therefore, the new LBA scheme is convergent, since it consistent and zero-stable.

3.5. Region of Absolute Stability (RAS) of LBA Scheme

To determine the regions of absolute stability of new LBA scheme, a method that requires neither the computation of roots of a polynomial nor solving of simultaneous inequalities was adopted. This method is called the Boundary Locus Method (BLM). The boundary locus method was used to obtained the stability polynomial of LBA scheme as

$$\bar{h}(\pi) = \left\{ \begin{aligned} & \left(-\frac{3151}{41343750000} \pi^7 - \frac{10411663}{20003760000000} \pi^6 \right) h^7 + \left(-\frac{5358324793}{41343750000} \pi^6 + \frac{16453}{27562500000} \pi^7 \right) h^6 \\ & + \left(\frac{54159020459}{600112800000000} \pi^6 + \frac{893657}{74418750000} \pi^7 \right) h^5 + \left(\frac{611966338171}{240045120000000} \pi^6 - \frac{485227}{2205000000} \pi^7 \right) h^4 + \\ & \left(\frac{11944063}{84672000000} \pi^6 - \frac{1577}{1323000000} \pi^7 \right) h^3 + \left(-\frac{13697597}{483840000} \pi^6 + \frac{529391}{17640000} \pi^7 \right) h^2 + \left(-\frac{3}{10} \pi^6 - \frac{3}{10} \pi^7 \right) h - \pi^6 + \pi^7 \end{aligned} \right\} \quad (3.7)$$

Using the stability polynomial in equation (3.7) on Matlab R2024a, we obtained the RAS of LBA scheme as

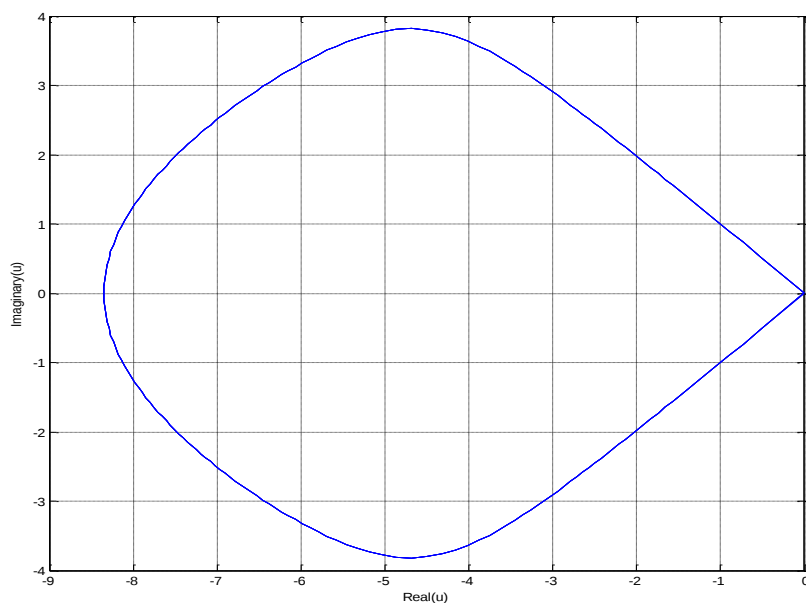


Figure 3.1. Regions of absolute stability of LBA scheme

4. Numerical Simulation

The new LBA scheme was applied on Volterra integro-differential equations of second kind and the results are numerically tabulated and textually shown to compare the results with the existing methods.

Example 4.1

Consider the Volterra integro-differential equation of the form

$$\rho'(\xi) = -\sin(\xi) - \cos(\xi) + \int_0^\xi 2 \cos(\xi - \tau) \rho(\tau) d\tau, \rho(0) = 1, 0 \leq \xi \leq 5 \quad (4.1)$$

with exact solution given as

$$\rho(\xi) = \exp(-\xi) \quad (4.2)$$

Source: See [Chen & Zhang, (2011); Faires & Burden, (2015); Majid & Mohamed, (2019)]

Example 4.2

Consider the Volterra integro-differential equation of the form

$$\rho'(\xi) = \xi \exp(1 - \rho(\xi)) - \frac{1}{(1 + \xi)^2} - \xi - \int_0^\xi \frac{1}{(1 + \tau)^2} \exp(1 - \rho(\xi)) d\tau, \rho(0) = 1 \quad (4.3)$$

with exact solution given by

$$\rho(\xi) = \frac{1}{1 + \xi} \quad (4.4)$$

Source See: [Faires & Burden, (2015); Majid & Mohamed, (2019)]

Example 4.3

Consider the nonlinear Volterra integro-differential equation of the form

$$\rho''(\xi) = \left(\frac{1}{2} \exp(-\xi) \sin(\xi) - \sin(\xi) \right) + \int_0^\xi \exp(\tau) \sin(\xi) \rho'(\tau) d\tau + \rho(\xi), \rho(0) = -1, \rho'(0) = 1 \quad (4.5)$$

with exact solution given by

$$\rho(\xi) = \sin(\xi) - \cos(\xi) \quad (4.6)$$

Source See: [Kamoh *et al.* (2017)]

The following Acronyms were used in the tables and figures.

Acronyms	Meaning
ξ	Points of Evaluation
ES	Exact Solution
CS	Computed Solution in LBA
ELBA	Error in LBA
EABM5	Error in Fifth Order Adams-Bashforth-Moulton Predictor-Corrector Method of Faires & Burden (2015)
E2P3B	Error in Two Point Three-Step Block Method as in This Research of Majid & Mohamed, (2019)
EGBDF5	Error in Combination of Boundary Value Methods and Fifth Order Generalized Backward Differentiation Formula by Chen & Zhang, (2011)
ETR	Error in Trapezoidal Rule using Three Quadrature Rules of Kamoh <i>et al.</i> , (2017)
EGR	Error in Gaussian Rule using Three Quadrature Rules of Kamoh <i>et al.</i> , (2017)
ES13R	Error in Simpsons 1/3 Rule using Three Quadrature Rules of Kamoh <i>et al.</i> , (2017)

Table 4.1. Comparison Numerical Results of Example 4.1

ξ	ES	CS	ELBA	EGBDF5	EABM5	E2P3B
0.25	0.77880078307140486825	0.77880078307140486820	2.9820e-17	2.3922e-02	8.1337e-03	6.1138 e-03
0.125	0.88249690258459540286	0.88249690258459540286	1.0000e-20	3.1790e-04	4.7616e-04	3.9009e-04
0.0625	0.93941306281347578612	0.93941306281347578612	0.0000e+00	4.3708e-06	2.1034e-06	1.6881e-05
0.03125	0.96923323447634408185	0.96923323447634408185	0.0000e+00	7.5567e-08	7.8509e-07	6.1208e-07
0.015625	0.98449643700540840599	0.98449643700540840599	0.0000e+00	-	2.6828e-08	2.0516e-08
0.0078125	0.99221793826024351211	0.99221793826024351211	0.0000e+00	-	8.7684e-10	6.6334e-10

Table 4.2. Comparison Numerical Results of Example 4.2

ξ	ES	CS	ELBA	EABM5	E2P3B
0.025	0.97560975609756097561	0.97560975609756028764	6.8797e-16	1.7212e-08	8.3237e-08
0.0125	0.98765432098765432099	0.98765432098765423863	8.2360e-17	2.0551e-09	3.8384e-09
0.00625	0.99378881987577639752	0.99378881987577639193	5.5900e-18	1.9089e-10	2.0775e-10
0.003125	0.99688473520249221184	0.99688473520249221546	3.5600e-18	1.1926e-11	1.2654e-11
0.0015625	0.99843993759750390016	0.99843993759750390120	1.0400e-18	7.4529e-13	9.6889e-13
0.00078125	0.99921935987509758002	0.99921935987509758099	9.7000e-19	4.6518e-14	4.3676e-13

Table 4.3. Comparison Numerical Results of example 4.3

ξ	ES	CS	ELBA6	ETR	EGR	ES13R
0.16	-0.82790907676138098573	-0.82790976676138098591	6.9000e-07	1.9000e-09	1.2512e-05	3.4135e-06
0.32	-0.63466885746632310092	-0.63466888990974523015	3.2443e-08	1.3900e-08	3.2919e-04	2.6456e-05
0.48	-0.42521574723780130526	-0.42521574489126308478	2.3465e-09	5.6600e-08	2.0860e-03	8.5434e-05
0.64	-0.20490031652190056171	-0.20490031092634578321	5.5956e-09	1.5330e-07	7.7089e-03	1.9242e-04
0.80	0.02064938155235734071	0.02064938023676800153	1.3156e-09	3.2662e-07	2.1169e-02	3.5240e-04
0.96	0.24567158222854160950	0.24567158258416091142	3.5562e-10	5.9210e-07	4.8093e-02	5.6512e-04

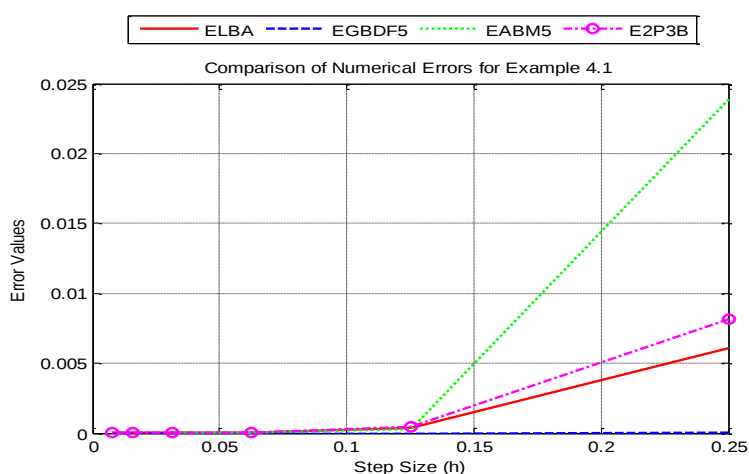


Figure 4.1. Graphical Curve of table 4.1

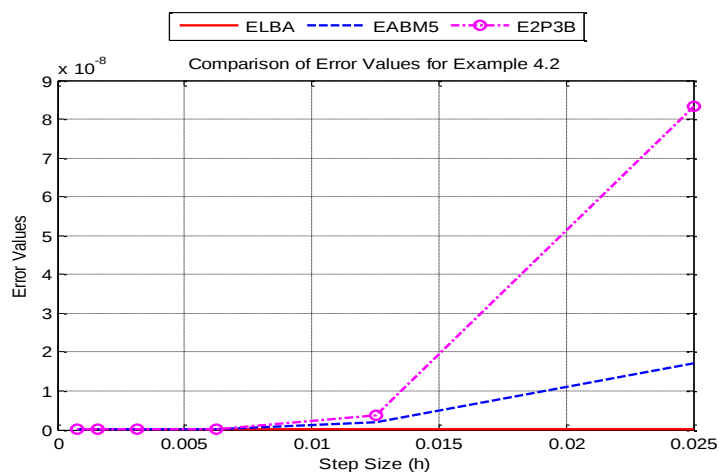


Figure 4.2. Graphical Curve of table 4.2

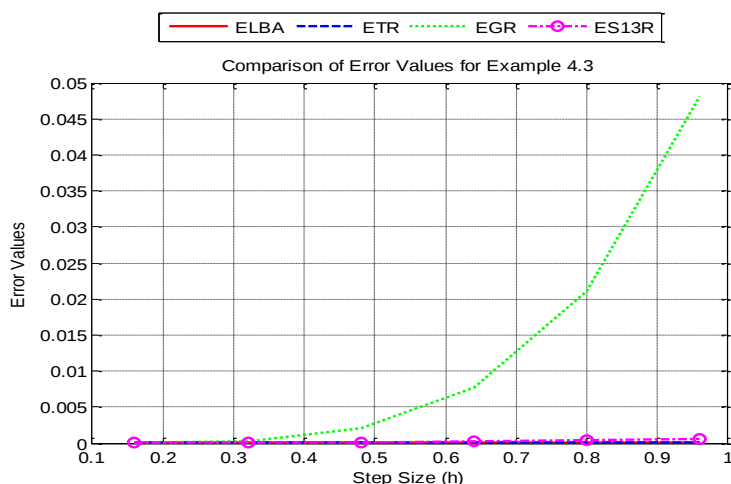


Figure 4.3. Graphical Curve of table 4.3

5. Discussion of Results

The derivation of the new third-derivative Linear Block Algorithm (LBA) begins by applying Proposition 3.1, which links the general linear multistep framework to the underlying block hybrid formulation. After deriving the scheme, its numerical properties are analyzed: using Corollaries 3.1 and 3.2, the scheme is shown to have a uniform order of eight, satisfying consistency requirements; zero-stability is verified by ensuring the roots of the characteristic polynomial lie within the unit disk; and convergence follows directly from Dahlquist's theorem. Finally, the region of absolute stability is determined using the Boundary Locus Method and MATLAB visualization confirms that the new LBA scheme possesses a robust stability region suitable for reliable numerical integration.

The numerical results presented in Tables 4.1 to 4.3, together with Figures 4.1 to 4.3, provide a clear comparative assessment of several numerical schemes applied to different Volterra integro-differential equations. For Example 4.1, the results show that the Linear Block Approach (LBA) achieves excellent agreement with the exact solution across all step sizes, as reflected in the extremely small ELBA values. The Generalized Backward Differentiation Formula (EGBDF5) also performs remarkably well with errors close to machine precision. In contrast, the Adams-Bashforth-Moulton method (EABM5) and the Two-Point Three-Step Block Method (E2P3B) display higher error values, particularly for relatively large step sizes, although both methods exhibit consistent reductions in error as the step size decreases. The plotted curve in Figure 4.1 highlights this trend, with the LBA and EGBDF5 curves nearly overlapping with the exact solution, while the other schemes converge more gradually.

In Example 4.2, the numerical comparison further reinforces the robustness of the LBA. The ELBA values remain extremely small across all step sizes, approaching the level of round-off error. The competing methods, EABM5 and E2P3B, show larger but steadily declining error values as the step size decreases, confirming their convergence but at a slower rate compared to the LBA. The closeness between the computed LBA solution and the exact solution (CS and ES) underscores the suitability of the block method for mild kernel Volterra equations. Figure 4.2 clearly illustrates this superior performance, with the LBA curve essentially indistinguishable from the exact solution, while EABM5 and E2P3B track below with visibly larger deviations, especially at coarser step sizes.

For Example 4.3, which involves a nonlinear Volterra integro-differential equation, the trend remains consistent but with more pronounced differences among the methods. The ELBA errors demonstrate high accuracy of the optimized block approach, although the increasing nonlinearity leads to slightly larger error magnitudes compared to Examples 4.1 and 4.2. The trapezoidal rule (ETR), Gaussian rule (EGR) and Simpson's 1/3 rule (ES13R), applied using three-point quadrature, exhibit significantly larger error levels, particularly EGR and ES13R, whose errors rise with step size due to their sensitivity to nonlinear integrands. Figure 4.3 visually confirms these disparities: the LBA curve remains closest to the exact nonlinear solution, while the classical quadrature-based methods diverge more noticeably. Overall, the tables and figures demonstrate the superior stability, accuracy, and convergence properties of the LBA across both linear and nonlinear Volterra integro-differential equations.

6. Summary and Conclusion

This study focused on the development and analysis of a new third-derivative Linear Block Algorithm (LBA) for solving Volterra integro-differential equations of the second kind. The method was derived using a generalized linear multistep framework, supported by interpolation, collocation, and the use of a Computer Algebra System (CAS) to obtain the continuous and discrete forms of the numerical scheme. Its fundamental numerical properties order and error constant, consistency, zero-stability, convergence, and region of absolute stability were rigorously established. The method was shown to possess a uniform order of eight and a stable characteristic polynomial, confirming strong theoretical reliability. The region of absolute stability was obtained using the Boundary Locus Method and visualized with MATLAB to further validate the method's robustness.

The numerical experiments conducted using three benchmark Volterra integro-differential equations demonstrated that the new LBA scheme consistently outperforms existing numerical methods, including Adams- Bashforth-Moulton predictor-corrector methods, generalized backward differentiation formulas, block multistep schemes, and standard quadrature rules. Results showed extremely small errors-often approaching machine precision—and excellent agreement with the exact solutions for both linear and nonlinear problems. Overall, the newly developed third-derivative LBA is efficient, accurate, and stable, making it a highly reliable tool for solving Volterra integro-differential equations of the second kind. Its superior convergence and stability properties position it as a valuable contribution to numerical analysis and computational mathematics.

7. Suggestion for Further Research

- 1) Optimized High-Order Linear Block Algorithms for Solving Volterra Integro-Differential Equations.
- 2) Adaptive Third-Derivative Block Methods for Nonlinear and Memory-Dependent Volterra Integro-Differential Systems.
- 3) Linear block algorithm for numerical solution of Fredholm integral equations.
- 4) Numerical Scheme for the solutions of Volterra Fredholm-integral Equations.
- 5) Derivation of block hybrid method for solving Volterra-Fredholm Integro-Differential Equations.

Declarations

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Competing Interests Statement

The author declares that he has no competing interests related to this work.

Consent for publication

The author declares that he consented to the publication of this study.

Authors' contributions

Author's independent contribution.

Availability of data and materials

Supplementary information is available from the author upon reasonable request.

Institutional Review Board Statement

Not applicable for this study.

Informed Consent

Not applicable for this study.

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